Robust PID-Controller Design Meeting Pole Location and Gain/Phase Margin Requirements for Time Delay Systems

Entwurf robuster PID-Regler mit Anforderungen an die Pollagen und an die Amplituden-/Phasenreserve für Totzeitsysteme

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A robust PID tuning method based on the parameter space approach is presented. In two steps a robust PID controller setting is designed which simultaneously satisfies pole location and gain/phase margin criteria. The plant model may be an arbitrary LTI time delay system with uncertain parameters (including the time delay), which are restricted by an operation domain.

Dieser Aufsatz stellt ein Verfahren zur Auslegung robuster PID-Regler vor, welches auf dem Parameterraumverfahren beruht. In zwei Schritten wird eine robuste PID-Reglerstellung entworfen, welche gleichzeitig Anforderungen an die Pollagen und die Amplituden-/Phasenreserve erfüllt. Als Streckenmodell kann ein beliebiges LTI Totzeitsystem mit unsicheren Parametern dienen, für welche ein Arbeitsbereich definiert wird.

**Keywords:** PID control, time delay, robust control, pole assignment, gain margin, phase margin

**Schlagwörter:** PID-Regularung, Totzeitsysteme, robuster Regelung, Polplatzierung, Amplitudenreserve, Phasenreserve

1 Introduction

The PID controller is the most common controller in industrial practice. But many surveys show that a high percentage of PID control systems are not tuned optimally (e.g. [7–9]), even though a great variety of different tuning methods exists. One major reason may be that so far there is no general tuning method, which leads for each application and for all essential specifications to a satisfying closed loop [3; 10; 14].

The main limitations of today’s tuning methods are, that often the controller design methods rely on restrictions on the plant model (concerning model order, pole and zero locations), that they are restricted to delay free systems, that they do not support essential specifications and that often robustness requirements (especially in parametric form) can not be incorporated in the design process.

To overcome some of the limitations, this paper extends the robust PID tuning method based on the parameter space approach [1] to cope with amplitude and gain margin specifications, additionally to certain pole location specifications. The problem of designing a PID controller $C(s) = (K_i + K_p s + K_d s^2)/(s (1 + T_R s))$ is considered, which satisfies the specifications for an arbitrary LTI time delay system $G(s) = A(s, q)/R(s, q) e^{-sL}$ with uncertain parameters $L$ and $q$ inside an operation domain $Q$.

The parameter space approach consists of two main steps. In the controller synthesis step several representatives $(L^*, q^*)^T$ are chosen out of the operation domain $Q$ (usually the vertices). For each representative and for each specification, the region in the space of controller parameters $k$ is computed, which fulfills the specification. After intersecting all the regions, a candidate for a robust controller $k^*$ is chosen from the intersection.

This controller satisfies the specifications for the representatives. The second step, the control loop analysis, is applied to ensure robust stability for the continuum of all values in $Q$. Now the regions in the space of plant parameters $(L, q)^T$ are computed, which fulfill each specification, while the controller parameters are fixed $k = k^*$. If $Q$ lies...
entirely in the intersection of the regions, then a solution of the problem is found. Note that the parameter space approach is particularly suitable, if there are no more than three uncertain plant parameters, so the regions can be visualised in a three-dimensional space.

An important subproblem is the calculation of the set of all stabilising PID controller parameters for an arbitrary LTI plant. It can be shown by different methods that for a fixed $K_p$ the stable region in the $(K_D, K_I)$-plane consists of convex polygons: In [17] a modified Hermite-Biehler-theorem is used, in [18] the Nyquist criterion is applied by calculating the real axis intersections of the Nyquist plot (which so far is only described for delay free systems) and in [2; 4; 5] a method based on the D-Decomposition is presented. A key question is the determination of the interval of $K_p$ which leads to stable polygons in the $(K_D, K_I)$-plane. Two different necessary conditions are presented in [18] and [5] where only the second paper extends to the delay case and gives an outlook to a necessary and sufficient condition (see example 4).

To meet gain and phase margin specifications, in [16] the margin borders are mapped into controller parameter planes of $(K_I, K_p)$ while gridding $K_D$. The mapping is based on the crossing of the Nyquist plot through the limit points $(-1/A_R)$ in case of a specified gain margin $A_R$, or $(-\cos(a_R) - j\sin(a_R))$ in case of a specified phase margin $\alpha_R$, respectively. There is not mentioned that this approach relays on monotony properties of the open loop frequency response. Alternatively, [13] maps the searched region in the $(K_I, K_D)$-plane while gridding $K_D$ by cutting of the parts of the stable polygon, whose corresponding Nyquist plots intersect the real axis, or the unit circle, respectively, on the ‘wrong’ side of the limit points. These cutting conditions lead to frequency dependent lines with slope $1/\omega^2$ in the $(K_I, K_D)$-plane. However, the monotony of the axis intercepts of the lines is obviously not proven, so the frequency has to be gridded at least in some intervals.

In this work, the D-Decomposition approach is used both in the controller synthesis and the control loop analysis step to calculate the set of stabilising controller and plant parameters. All other specifications are reduced to the basic stability case. In [11; 12] this approach was used with $\sigma$-stability (all poles have a real part smaller than a real number $\sigma_0$). The stated problem of designing a robust PID controller is to find a set of controller parameters $k = k^*$, such that the closed loop meets the following specifications for all values of $(L, q)^T \in Q$:

- Hurwitz stability (asymptotic stability),
- $\sigma$-stability: all roots have a real part smaller than a real number $\sigma_0$,
- Gain margin larger than $A_R$,
- Phase margin larger than $\alpha_R$.

3 Hurwitz Stability

First the specification Hurwitz stability is treated. Then it is shown that all other specifications can be reduced to the Hurwitz case.

The characteristic function of the loop in Fig. 1

$$P(s, k, L, q) = (K_I + K_P s + K_D s^2) A(s, q) + s (1 + T_R s) R(s, q) e^{sL},$$

with polynomials

$$A(s, q) = a_0(q) s + \ldots + a_m(q) s^m,$$  

$$B(s, q) = b_0(q) s + \ldots + b_n(q) s^n,$$

belongs to the class of quasipolynomials due to the time delay\(^1\) (with $a_m(q) \neq 0$, $b_n(q) \neq 0$).

The principal term condition [15] requires for Hurwitz stability that in the case $K_D \neq 0$ the degrees fulfill $n \geq m + 2$. In the sequel we treat only this case (i.e. we assume a proper $A(s, q)/R(s, q)$ for $T_R \neq 0$).

\(^1\) Note that $b_0(q) = 0$ for the basic case of a PID controller (1). Later a $b_0(q) \neq 0$ may appear through transformations, see Sect. 4.1.
The calculation of a Hurwitz stable region in a parameter space (either \( k \) or \((L, q)^T\)) by the D- Decomposition method is based on the fact that the roots of the quasipolynomial (4) with continuous coefficient functions \( a_i(q), b_i(q) \) move continuously when the parameters are changed continuously. Thus, a stable quasipolynomial, whose roots all lie in the left half plane (LHP), becomes unstable if and only if at least one root crosses the imaginary axis. The corresponding parameter values of the root crossings form the stability boundaries in the parameter space, which can be classified into three cases: the real root boundary (RRB), where a root crosses the imaginary axes at the origin (substitute \( s = 0 \) in the quasipolynomial), the infinite root boundary (IRB), where a root leaves the LHP at infinity (set \( |s| \rightarrow \infty \) and the complex root boundary (CRB), where a pair of conjugate complex roots crosses the imaginary axes (substitute \( s = jo \) and sweep over all real \( \omega > 0 \)). These stability boundaries separate different regions in the parameter space. To classify a region as Hurwitz stable it suffices to prove stability for one inner test point (e.g. by the Nyquist criterion).

### 3.1 Controller Synthesis

For each fixed representative \((L^*, q^*)^T\) of the plant parameters of the Hurwitz stability boundaries of (4) in the controller parameter \(K\)-space are searched. The RRB is given by the equation

\[
P(0, k) = K_I A(0) + B(0) = 0 \Leftrightarrow K_I = -\frac{b_0}{a_0} . \tag{7}
\]

Quasipolynomials possess an infinite number of roots, which cannot be calculated analytically in the general case. However, the asymptotic location of roots far from the origin is well known [6], which may lead to IRBs. It can be shown that IRBs only exist, if the degree equation \( n = m + 2 \) is fulfilled (in case of \( K_D \neq 0 \)) (for details see [11]). In this case, the IRBs can be described by the following equation

\[
K_D = \pm \frac{b_n}{a_m} . \tag{8}
\]

To determine the CRBs, the root condition \( P(jo, k) = 0 \) can be separated into a system of two equations for real and imaginary part

\[
\begin{bmatrix}
P_R(jo, k) \\
P_I(jo, k)
\end{bmatrix} = \begin{bmatrix} R_A & -\omega^2 R_A \\
I_A & -\omega^2 I_A
\end{bmatrix} \begin{bmatrix} K_I \\
K_D
\end{bmatrix} + \begin{bmatrix} R_B - K_P o I_A \\
I_B + K_P o R_A
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}, \tag{9}
\]

where \( B(jo) = B(jo) e^{joL^*} \) and \( R, I \) denote the real and imaginary parts of \( A, B \) and \( P \) at \( (jo) \).

Clearly, the matrix multiplying \((K_I, K_D)^T\) is singular. Thus, the key idea is to fix \( K_P = K_P^* \) and to evaluate the CRB in the \((K_D, K_I)^\) plane. A solution of (9) exists and only exists for the real zeros \( \omega_g \) of

\[
g(\omega) = \det \begin{bmatrix} R_A & R_B - K_P o I_A \\
I_A & I_B + K_P o R_A
\end{bmatrix} = \omega K_P^* \left( R_A^2 + I_A^2 \right) + R_A I_B - I_A R_B . \tag{10}
\]

The zeros of \( g(\omega) \) are called singular frequencies. For each frequency a straight line results as CRB in the \((K_D, K_I)^\) plane, ruled by the equation

\[
K_I = \alpha^2 K_D + K_0^*(\omega_0), \tag{11}
\]

where \( K_0^*(\omega_0) \) can be determined from the first or second row of (9).

The singular frequencies may be determined by a graph of

\[
K_P(\omega) = -\frac{R_A I_B - I_A R_B}{\omega \left( R_A^2 + I_A^2 \right)} . \tag{12}
\]

Graphically, the singular frequencies for a fixed \( K_P^* \) are the abscissa values of the intersections between the \( K_P(\omega) \)-plot and the \((K_D, K_I)^\) -line. Due to the time delay the number of singular frequencies is infinite. This corresponds to the fact that the Nyquist plot of a time delay system encircles the origin infinitely.

Thus, the stability boundaries RRB, IRB and CRB are straight lines in the \((K_D, K_I)^\) plane and partition the plane into convex polygons. Additionally, for each boundary line the ‘stable’ side can be determined: If the sign of \( \partial K_P(\omega)/\partial \omega \) at the corresponding singular frequency \( \omega_0 \) is positive, the line side showing to decreasing \( K_I \)-values or increasing \( K_D \)-values possesses the higher number of stable poles (see [5]).

Two questions remain: How many singular frequencies have to be evaluated and which \( K_P \)-values lead to stable polygons?

#### High frequency behaviour

For high frequencies, function (12) tends to a much simpler limit function of the form

\[
K_P(\omega) \xrightarrow{\omega \rightarrow \infty} \alpha \omega^\beta \text{ trig}(\omega L^*), \tag{13}
\]

where \( \alpha \) and \( \beta \) are real constants (\( \beta \geq 1 \)) and \text{ trig} is either the sin or cos function. It can be shown (see [11]), that singular frequencies which lie in a frequency range where (13) and (12) are practically identical result in CRB lines which do not contribute to boundaries of the stable polygon. So, it suffices to compute and evaluate only a finite number of low singular frequencies, given by the comparison of (13) and (12).

#### Stabilising \( K_P \)-interval

The question of determining the interval of \( K_P \) which leads to stable polygons is analytically solved so far only for a first order time delay system [17]. For higher order systems the following procedure is used, which is based on...
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Figure 2: Function \( K_P(\omega) \) (solid), its limit function (dashed), singular frequencies for \( K_P^* = 1 \) (×-marks) and stabilising \( K_P \)-interval (dash-dotted) of \( G_1(s) \).

the necessary stability condition of theorem 4 in [5]. Note that an infinite number of conditions has to be checked in the case of time delay systems to apply the theorem completely. However, the theorem motivates the following procedure:

Observe that depending on the value of \( K_P^* \) a different number of singular frequencies exists. The theorem states, that the number of singular frequencies (up to a certain system dependent limit frequency) has to exceed a certain (system dependent) number. So, the first interval to use is the interval which produces the highest number of singular frequencies (see dash-dotted lines in Fig. 2). If the resulting stable polygon at the borders of the \( K_P \)-intervals does not close, the adjacent \( K_P \)-interval with a lower number of singular frequencies has to be added to the interval of stabilising \( K_P \), and so on.

3.2 Example

The function \( K_P(\omega) \) and the resulting boundaries in the \((K_D, K_I)\)-plane for a fixed \( K_P^* = 1 \) are demonstrated in Figs. 2 and 3 for the example system (controlled by ideal PID \( T_R = 0 \))

\[
G_1(s) = \frac{1}{s+1} e^{-0.7t}. \tag{14}
\]

Stability checks of test points prove that only the polygon around \( P_1 \) is stable. The entire stable region in the \( k \)-space can be computed by gridding \( K_P \) in its stabilising interval and extracting the stable polygon for each gridded \( K_P^* \) (see Fig. 4).

3.3 Control Loop Analysis

Now the Hurwitz stable region in the plant parameter space with fixed controller parameters \( k^* \) is determined by mapping the stability boundaries of the quasipolynomial

\[
P(s, L, q) = D(s, q) + B(s, q) e^{L}, \tag{15}
\]

where

\[
D(s, q) = \left( K_P^* + K_P^* s + K_D^* s^2 \right) A(s, q)
= d_0(q) + d_1(q) s + \ldots + d_{m'}(q) s^{m'} \tag{16}
\]

The case of mapping the boundaries to a plane of \( L \) and one additional parameter \( q \) is treated. If there are more uncertain parameters, then they have to be gridded.

The RRB are lines \( q = q^* \), where \( q^* \) are the solutions of the equation

\[
d_0(q) + b_0(q) = 0. \tag{17}
\]

If the degrees of \( B(s, q) \) and \( D(s, q) \) fulfill \( n = m' \), then the solutions of the following equation lead to IRB

\[
d_m(q) \pm b_n(q) = 0. \tag{18}
\]
The CRB condition $P(j\omega, L, q) = 0$ can be split into a modulus and a phase equation

$$\left| \frac{D(j\omega, q)}{B(j\omega, q)} \right| = 1, \tag{19}$$

$$L(\omega, q, k) = \frac{1}{\omega} (\arg D(j\omega, q) - \arg B(j\omega, q) + (2k - 1)\pi), \quad k \in \mathbb{Z}. \tag{20}$$

The modulus equation does not depend on $L$ and can be transformed to a polynomial in $\omega^2$. Thus, the analysis step can be reduced to the Hurwitz case, if $q$ the second parameter.

In that case, the transformations are

Concerning the synthesis step, this case can be reduced to the Hurwitz case by transformations of the polynomials $A$ and $B$ and the parameters. By the help of the transformations the requirements can be met using the methods presented in the previous sections.

### 4 Reducing Other Specifications to the Hurwitz Case

In this section $\sigma$-stability and gain and phase margin requirements are reduced to the Hurwitz stability case by transformations of the polynomials $A$ and $B$ and the parameters. By the help of the transformations the requirements can be met using the methods presented in the previous sections.

#### 4.1 $\sigma$-stability

The requirement $\sigma$-stability defines an upper limit $\sigma_0$ on the real part of all closed loop poles. For instance, this requirement can be used to limit the settling time of the closed loop.

Concerning the synthesis step, this case can be reduced to the Hurwitz case by substituting $s = v + \sigma_0$. It can be shown that following transformations result

$$K'_{\theta} = K_{\theta} + K_{D} \sigma_0 + K_{D} \sigma_0^2,$$

$$K'_{P} = K_{P} + 2K_{D} \sigma_0,$$

$$A'(v) = A(v + \sigma_0),$$

$$B'(v) = e^{\sigma_0 L} B(v + \sigma_0). \tag{21}$$

The analysis step can be reduced to the Hurwitz case, if the second parameter $q$ enters in form of a dc-gain into the quasipolynomial. In that case, the transformations are

$$q'' = q e^{\sigma_0 L}, \quad D''(v) = D(v + \sigma_0), \tag{22}$$

$$L'' = L, \quad B''(v) = B(v + \sigma_0).$$

#### 4.2 Gain and Phase Margin

Gain and phase margin specifications are treated by mapping the crossing of the open loop Nyquist plot through a limit point in the complex plain. Each limit gain and phase margin $\zeta$ corresponds to a limit point $\rho + j\zeta$, where $\rho = -1/A_R$, $\zeta = 0$ in the case of a required gain margin $A_R$, and $\rho = -\cos(\alpha_R)$, $\zeta = -\sin(\alpha_R)$ in the case of a required phase margin $\alpha_R$ (see [16]). Note that this approach relies on certain conditions with respect to monotony properties of the frequency response, which have to be studied further.

The open loop frequency response is

$$F(j\omega, k) = \frac{(K_I + K_P j\omega + K_D (j\omega)^2)}{B(j\omega)} e^{-\rho j L} \tag{23}$$

The crossing condition is $F(j\omega, k) = \rho + j\zeta$, which leads to

$$(K_I + K_P (j\omega) + K_D (j\omega)^2) A(j\omega) + \hat{B}(j\omega) e^{\rho j L} = 0, \tag{24}$$

where

$$\hat{B}(j\omega) = -B(j\omega) (\rho + j\zeta). \tag{25}$$

By applying the transformation (25) the crossing condition reduces to the Hurwitz case (4). Note that for $\rho = -1$, $\zeta = 0$ (critical point in the Nyquist plot) the transformation is the identity.

### 5 Illustrative Example

The following example illustrates the proposed method. A robust PID controller (in ideal form $T_R = 0$) for a first order time delay system with uncertainty in the delay value

$$G_2(s) = \frac{K}{1 + s} e^{-s L}, \quad Q : \ L \in [0.7; 1.3], \ K = 1 \tag{26}$$

is searched, which satisfies following specifications over the operation domain $Q$:

- Hurwitz stability,
- $\sigma$-stability with $\sigma_0 = -0.2$,
- Gain margin larger than $A_R = 1.96$,
- Phase margin larger than $\alpha_R = 45^\circ$.

![Figure 5: Regions of controller parameters $k$ for system $G_2(s)$, which satisfy each specification for both representatives $L = 0.7$ and $L = 1.3$.](image)
Figure 5 shows the result of the controller synthesis step: For each specification the region of controller parameters \( k \) is computed as the intersection of the corresponding regions for the representatives \( L = 0.7 \) and \( L = 1.3 \).

After intersecting the four regions of Fig. 5, the controller parameters \( k^* \) (\( K_I = 0.58 \), \( K_P = 0.82 \), \( K_D = 0.26 \)) are chosen. For the control loop analysis the loop gain \( K \) is used as a second free parameter. The result with respect to Hurwitz stability can be seen in Fig. 6. Repeating the analysis step for the other specifications and extracting the regions, which fulfill the specifications, leads to Fig. 7.

Clearly, the operation domain \( Q \) lies entirely in the intersection of all regions, so the designed controller \( k^* \) meets all specifications robustly. A plot of the closed loop responses to a unit step command at different values of \( L \) can be seen in Fig. 8.

### 6 Conclusions and Outlook

The parameter space approach applied to the design of robust PID controllers for uncertain LTI time delay systems is extended to meet Hurwitz, \( \sigma \)-stability and gain and phase margin specifications. A main result of this paper is, that gain and phase margin specifications can be reduced to the Hurwitz case. This leads to the important result, that the region in the PID controller parameter space, which fulfill a certain minimum gain or phase margin, consists of simple convex polygons in the \((K_D, K_I)\)-plane.

Consequently the proposed design method can be applied yet both to a large variety of systems (all linear SISO systems) and to a variety of specifications (pole location and frequency response requirements), which meets the limitations of existing tuning methods.

In future work the list of specifications has to be further extended. Also the required monotony properties of the open loop frequency response must be described precisely. Additionally, the development of an interactive graphical software package seems very promising to be a helpful tool in daily control engineer’s work.

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